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NONLINEAR ANALYSIS OF LAMINATED AXISYMMETRIC SPHERICAL SHELLS

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Abstract—The governing differential equations for the problem of laminated axisymmetric spherical shells undergoing large deformations are formulated using the principle of virtual work. An analytical solution of the governing equations based on the Chebyshev–Galerkin spectral method is investigated. The efficacy and applicability of the solution procedure is discovered using numerical results. Parametric studies are conducted to bring out the effect of factors like orthotropy ratio, R/h ratio, shear deformation and opening angle on the large deflection behaviour of laminated orthotropic spherical shells and interesting observations are made. The numerical results should prove helpful in testing the nonlinear composite shell finite elements.

INTRODUCTION

The use of laminated anisotropic structural configurations is increasing in applications with aerospace, missile, hydrospace and auto industries owing to the variety of advantages they offer when compared to the metallic materials. There are possibilities that these elements are called upon to perform under severe loading conditions causing large deformations. The number of investigations dealing with the behaviour of laminated composite shells undergoing geometrically nonlinear deformations are rather limited in comparison to the investigations concerning the geometrically nonlinear behaviour of composite plates.

Most of the work in the area of large deformation analysis of laminated shells is a logical extension of works with composite plates or isotropic/orthotropic shell structures. The stiffness, anisotropic material properties, bending–stretching coupling complicate the analysis of shells made up of composite materials.

Librescu (1987), Stein (1986) and Dennis and Palazotto (1990) have contributed to the geometrically nonlinear theories of laminated composite shells. Noor and his co-workers (Noor and Mathers, 1974; Noor and Heartly, 1977; Noor and Anderson, 1982; Noor and Peters, 1986) have done a significant amount of research in the development and application of shell finite elements applied to geometrically nonlinear theories of laminated shear-flexible elements based on assumed strains for the nonlinear analysis of shells. Chang and Sawamiphakdi (1981) and Chao and Reddy (1984) have given the formulations of 3-D degenerated shell elements based on the total Lagrangian and updated Lagrangian descriptions respectively. Reddy and Chandrashekhara (1985) presented results for large deflections of laminated shell panels using a doubly curved shell element. Booton and Tennyson (1979), Sheinman and Simitses (1983) and Saigal *et al.* (1986) have used the finite element technique for the analysis of imperfect laminated composite shells. However, to the authors' knowledge, there is hardly any analytical solution for the nonlinear analysis of laminated spherical shells. The object of the present investigation is to give an analytical solution to the problem of large axisymmetric deformation of laminated orthotropic spherical shells, which will provide bench mark numerical results to test the accuracy of finite element solutions. In addition, the influence of different degrees of nonlinearity on the large deformation behaviour of laminated spherical shells has been studied. Results of a number of parametric studies are presented which will facilitate the practising engineers with a better understanding of large deformation behaviour of laminated spherical shells.

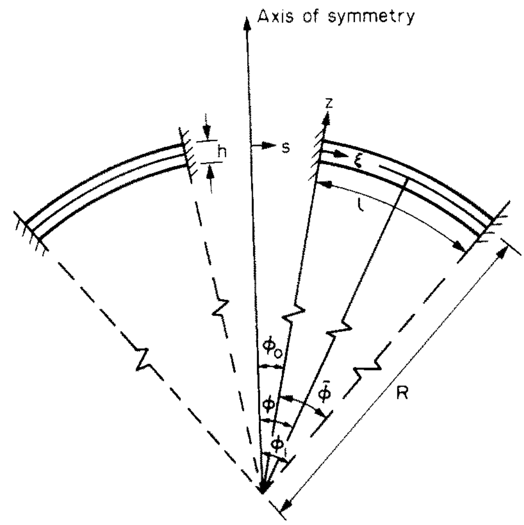

Fig. 1. Geometry of laminated annular shell.

MATHEMATICAL FORMULATION

An annular, moderately thick spherical shell composed of a number of perfectly bonded, linearly elastic orthotropic layers is considered. The principal material directions in each layer are assumed to coincide with the shell curvature directions. The coordinates s and θ are assumed to coincide with the same principal curvature directions of the shell reference surface. z is the normal outward distance from the reference surface (see Fig. 1).

STRAINS IN NONLINEAR ELASTICITY

The nonlinear strain displacement relations for a body undergoing large deformations in general curvilinear coordinates are given by Novozhilov (1961). In the spherical polar (ϕ–θ–r) coordinate system, Lamé's parameters are given by

$$H_1 = r, \quad H_2 = r\sin\phi, \quad H_3 = 1, \tag{1}$$

where a line element is given by

$$ds^2 = (H_1 d\phi)^2 + (H_2 d\theta)^2 + (H_3 dr)^2. \tag{2}$$

If the displacements in the ϕ, θ and r directions are respectively u, v and w, the nonlinear strains are then obtained as:

$$e_{\phi\phi} = \left(\frac{1}{r}\frac{\partial u}{\partial\phi} + \frac{w}{r}\right) + \frac{I_1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial\phi} - \frac{u}{r}\right)^2 + \frac{I_3}{2}\left\{\left(\frac{1}{r}\frac{\partial u}{\partial\phi} + \frac{w}{r}\right)^2 + \left(\frac{1}{r}\frac{\partial v}{\partial\phi}\right)^2\right\},$$

$$e_{\theta\theta} = \left(\frac{w}{r} + \frac{\cot\phi}{r}u + \frac{1}{r\sin\phi}\frac{\partial v}{\partial\theta}\right) + \frac{I_1}{2}\left(\frac{1}{r\sin\phi}\frac{\partial w}{\partial\theta} - \frac{v}{r}\right)^2$$

$$+ \frac{I_3}{2}\left\{\left(\frac{1}{r\sin\phi}\frac{\partial v}{\partial\theta} + \frac{w}{r} + \frac{\cot\phi}{r}u\right)^2 + \left(\frac{1}{r\sin\phi}\frac{\partial u}{\partial\theta} - \frac{\cot\phi}{r}v\right)^2\right\},$$

$$e_{rr} = \frac{\partial w}{\partial r} + \frac{I_3}{2}\left\{\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial v}{\partial r}\right)^2\right\},$$

$$e_{\phi r} = \left(\frac{\partial u}{\partial r} - \frac{u}{r} + \frac{1}{r}\frac{\partial w}{\partial\phi}\right) + I_2\left\{\frac{\partial w}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\phi} - \frac{u}{r}\right) + \frac{\partial u}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial\phi} + \frac{w}{r}\right)\right\} + I_3\frac{1}{r}\frac{\partial v}{\partial r}\frac{\partial v}{\partial\phi},$$

$$e_{r\theta} = \left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r\sin\phi}\frac{\partial w}{\partial \theta}\right) + I_2 \frac{\partial w}{\partial r}\left(\frac{1}{r\sin\phi}\frac{\partial w}{\partial \theta} - \frac{v}{R}\right)$$

$$+ I_2\left(\frac{1}{r\sin\phi}\frac{\partial v}{\partial \theta} + \frac{\cot\phi}{r}u + \frac{w}{r}\right)\frac{\partial v}{\partial r} + I_3\frac{\partial u}{\partial r}\left(\frac{1}{r\sin\phi}\frac{\partial u}{\partial \theta} - \frac{\cot\phi}{r}v\right),$$

$$e_{\phi\theta} = \left(\frac{1}{r}\frac{\partial v}{\partial \phi} - \frac{\cot\phi}{r}v + \frac{1}{r\sin\phi}\frac{\partial u}{\partial \theta}\right) + I_1\left(\frac{1}{r}\frac{\partial w}{\partial \phi} - \frac{u}{r}\right)\left(\frac{1}{r\sin\phi}\frac{\partial w}{\partial \theta} - \frac{v}{r}\right)$$

$$+ I_3\left\{\left(\frac{1}{r\sin\phi}\frac{\partial v}{\partial \theta} + \frac{\cot\phi}{r}u + \frac{w}{r}\right)\frac{1}{r}\frac{\partial v}{\partial \phi} + \left(\frac{w}{r} + \frac{1}{r}\frac{\partial u}{\partial \phi}\right)\left(\frac{1}{r\sin\phi}\frac{\partial u}{\partial \theta} - \frac{\cot\phi}{r}v\right),\right. \quad (3)$$

where I_1, I_2 and I_3 are coefficients taking values of either zero or one depending upon whether the associated nonlinear term is considered in the analysis or not. Three different approximations regarding the range of nonlinearities are considered in the present investigation:

(a) The nonlinear terms corresponding to the squares and products of the rotation of the tangents alone are considered (Von Karman type nonlinearity). In such a case, $I_1 = 1$, $I_2 = I_3 = 0$;

(b) The nonlinear terms due to the squares and products of the rotation of the tangents are considered along with the products of strains and rotations. Here $I_1 = I_2 = 1$ and $I_3 = 0$;

(c) All the nonlinear terms are considered. In this case, $I_1 = I_2 = I_3 = 1$.

All the strains can now be transformed to the shell coordinate system by setting

$$r = (R+z),$$
$$ds = R\,d\phi. \quad (4)$$

In the present analysis, only spherical shells undergoing moderately large axisymmetric deformations are considered. With the assumption that the strains are not so large that the squares and products of strains can be neglected compared to the products of strains and moderate rotations, the cases (a) and (b) alone are considered in the present analysis. The nonvanishing nonlinear strains for the axisymmetric case can then be written as:

$$e_z = e_r = \frac{\partial w}{\partial z},$$

$$e_s = e_{\phi\phi} = \frac{1}{(1+z/R)}\left(\frac{\partial u}{\partial s} + \frac{w}{R}\right) + \frac{I_1}{2(1+z/R)^2}\left(\frac{\partial w}{\partial s} - \frac{u}{R}\right)^2,$$

$$e_\theta = e_{\theta\theta} = \frac{1}{(1+z/R)}\left(\frac{\cot\phi}{R}u + \frac{w}{R}\right),$$

$$e_{sz} = e_{\phi z} = \frac{1}{(1+z/R)}\left(\frac{\partial w}{\partial s} - \frac{u}{R}\right) + \frac{\partial u}{\partial z} + \frac{I_2}{(1+z/R)}\frac{\partial u}{\partial z}\left(\frac{\partial u}{\partial s} + \frac{w}{R}\right). \quad (5)$$

DISPLACEMENT FIELDS

Assuming an axisymmetric displacement field allowing for shear deformation in the sense of a first order theory,

$$u(s, \theta, z) = \hat{u}(s) + z\hat{\alpha}(s),$$
$$w(s, \theta, z) = \hat{w}(s), \quad (6)$$

where $\hat{u}$ and $\hat{w}$ represent the displacements in the meridional and transverse directions at the point corresponding to the reference surface and $\hat{\alpha}$ is the rotation of the normal about the meridian.

SHELL STRAIN–DISPLACEMENT RELATIONS

Now substituting relations (6) in eqns (5), strains at any point (s, z) are written in terms of the strains and change in curvatures at the reference surface. With the assumption that $(h/R)^2 \ll 1$, the strains can be written as:

$$\begin{aligned} e_z &= 0, \\ e_s &= (e_s^0 + z\lambda_s^0)/(1+z/R), \\ e_\theta &= (e_\theta^0 + z\lambda_\theta^0)/(1+z/R), \\ e_{sz} &= e_{sz}^0/(1+z/R), \end{aligned} \tag{7}$$

where the reference surface strains are given by:

$$\begin{aligned} e_s^0 &= \frac{d\hat{u}}{ds} + \frac{\hat{w}}{R} + \frac{I_1}{2}\left(\frac{d\hat{w}}{ds} - \frac{\hat{u}}{R}\right)^2, \\ e_\theta^0 &= \frac{\cot\phi}{R}\hat{u} + \frac{\hat{w}}{R}, \\ e_{sz}^0 &= \left(\frac{d\hat{w}}{ds} + \hat{\alpha} - \frac{\hat{u}}{R}\right) + I_2\hat{\alpha}\left(\frac{d\hat{u}}{ds} + \frac{\hat{w}}{R}\right) \end{aligned} \tag{8}$$

and the reference surface curvatures are given by

$$\lambda_s^0 = \frac{d\hat{\alpha}}{ds} - I_1\frac{\hat{\alpha}}{R}\left(\frac{d\hat{w}}{ds} - \frac{\hat{u}}{R}\right) + \frac{I_1}{2R}\left(\frac{d\hat{w}}{ds} - \frac{\hat{u}}{R}\right)^2 \quad \text{and} \quad \lambda_\theta^0 = \cot\phi\frac{\hat{\alpha}}{R}. \tag{9}$$

In eqn (7) above, the term representing variation of transverse shear strain through the thickness has been neglected.

STRESS RESULTANTS AND STRESS COUPLES

Corresponding to the strains defined in eqns (7)–(9), the stress resultants and stress couples are defined as:

$$(N_s N_\theta Q_s M_s M_\theta) = \int_h (\sigma_s \sigma_\theta \tau_{sz} z\sigma_s z\sigma_\theta)(1+z/R)\,dz, \tag{10}$$

where h is the thickness of the shell.

CONSTITUTIVE RELATIONS

Assuming plane stress conditions in the $s\theta$ plane, the stress–strain relation for the kth layer of the shell bounded by the surfaces $z = h_k$ and $z = h_{k-1}$ are given by:

$$\begin{aligned} \sigma_s &= Q_{11}^k e_s + Q_{12}^k e_\theta, \\ \sigma_\theta &= Q_{12}^k e_s + Q_{22}^k e_\theta \quad \text{and} \quad \tau_{sz} = Q_{44}^k e_{sz}, \end{aligned} \tag{11}$$

where

$$Q_{11}^k = \frac{E_s^k}{(1-v_{s\theta}^k v_{\theta s}^k)}, \quad Q_{22}^k = \frac{E_\theta^k}{(1-v_{s\theta}^k v_{\theta s}^k)},$$

$$Q_{12}^k = \frac{v_{s\theta} E_s^k}{(1-v_{s\theta}^k v_{\theta s}^k)}, \quad Q_{44}^k = G_{sz}^k. \tag{12}$$

In the above relations E_s^k and E_θ^k are the Young's moduli of elasticity of the material in the kth layer in the meridional and circumferential directions, G_{sz}^k is the transverse shear modulus and v_{ij}^k $(i, j = s, \theta)$ are the Poisson's ratios.

STRESS RESULTANTS AND STRAIN RELATIONS

Substituting for the strains in eqn (11) from eqns (7)–(9) and using eqn (10), the relation between the stress resultants and reference surface strains are written as:

$$\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \{e\} \\ \{\lambda\} \end{Bmatrix} \tag{13}$$

and

$$Q_s = K^2 A_{44} e_{sz}^0. \tag{14}$$

In relations (13),

$$\{N\}^{\mathrm{T}} = \{N_s N_\theta\}, \quad \{e\}^{\mathrm{T}} = \{e_s^0 e_\theta^0\},$$

$$\{M\}^{\mathrm{T}} = \{M_s M_\theta\}, \quad \{\lambda\}^{\mathrm{T}} = \{\lambda_s^0 \lambda_\theta^0\}. \tag{15}$$

In eqns (13) and (14),

$$(A_{ij} B_{ij} D_{ij}) = \sum_{k=1}^{L} \int_{h_{k-1}}^{h_k} Q_{ij}(1 z z^2)\,\mathrm{d}z, \tag{16}$$

where L is the number of layers in the laminate. In (14), K^2 is the shear correction factor introduced to account for the nonuniform distribution of the shear strains through the thickness of the shell.

EQUATIONS OF EQUILIBRIUM AND BOUNDARY CONDITIONS

The equations of equilibrium and the associated boundary conditions for the laminated spherical shell are derived using the principle of virtual-work. Equating the algebraic sum of virtual work of all the forces acting on the shell to zero for an arbitrary virtual displacement,

$$\delta W_{\mathrm{I}} + \delta W_{\mathrm{E}} = 0, \tag{17}$$

where

W_{I} = virtual work done by internal forces,

W_{E} = virtual work done by external forces.

The Euler–Lagrange equations for axisymmetric large deformations of laminated spherical shells are then given by:

$$\frac{dN_s}{ds}+(N_s-N_\theta)\frac{\cot\phi}{R}+\frac{Q_s}{R}=\frac{I_1N_s}{R}\left(\frac{\hat{u}}{R}-\frac{d\hat{w}}{ds}\right)-\frac{I_1M_s}{R^2}\left(\frac{\hat{u}}{R}-\frac{d\hat{w}}{ds}-\hat{\alpha}\right)$$
$$-I_2Q_s\hat{\alpha}\cot\phi/R-I_2\frac{d}{ds}(Q_s\hat{\alpha}),$$

$$\frac{dM_s}{ds}+(M_s-M_\theta)\frac{\cot\phi}{R}-Q_S=\frac{I_1M_s}{R}\left(\frac{\hat{u}}{R}-\frac{d\hat{w}}{ds}\right)+I_2Q_s\left(\frac{d\hat{u}}{ds}+\frac{\hat{w}}{R}\right),$$

$$\frac{N_s+N_\theta}{R}-\frac{dQ_s}{ds}-\frac{Q_s\cot\phi}{R}=q-I_1\frac{d}{ds}\left(N_s\left(\frac{\hat{u}}{R}-\frac{d\hat{w}}{ds}\right)\right)$$
$$-I_1N_s\frac{\cot\phi}{R}\left(\frac{\hat{u}}{R}-\frac{d\hat{w}}{ds}\right)+I_1\frac{d}{ds}\left(\frac{M_s}{R}\left(\frac{\hat{u}}{R}-\frac{d\hat{w}}{ds}-\hat{\alpha}\right)\right)$$
$$+I_1M_s\frac{\cot\phi}{R^2}\left(\frac{\hat{u}}{R}-\frac{d\hat{w}}{ds}-\hat{\alpha}\right)-I_2Q_s\hat{\alpha}/R, \quad (18)$$

with the associated boundary conditions requiring that any one in each of the following pairs of quantities is to be prescribed at a circumferential edge:

$$\hat{u}=0 \quad \text{or} \quad N_s+I_2Q_s\hat{\alpha}=0,$$
$$\hat{\alpha}=0 \quad \text{or} \quad M_s=0,$$
$$\hat{w}=0 \quad \text{or} \quad Q_s+I_1\left\{(N_s-M_s/R)\left(\frac{d\hat{w}}{ds}-\frac{\hat{u}}{R}\right)-M_s\hat{\alpha}/R\right\}=0. \quad (19)$$

The stress–resultant–strain relations (13), (14), the equilibrium equations (18) and the boundary conditions given in (19), completely define the problem of large deflection behaviour of an axisymmetric laminated spherical shell. These equations are nonlinear in nature and the methodologies for the solution of these equations are discussed below.

LINEARIZATION OF EQUATIONS

There are several established techniques for linearizing the nonlinear differential equations such as the perturbation technique, the quasilinearization, the time-wise differentiation techniques, etc. In the present analysis, a Taylor series expansion procedure is used to linearize the equations. Alwar and Nath (1977) and Nath and Alwar (1978) have used this technique for solving the nonlinear equations of large deflection and buckling behaviour of circular plates and shallow spherical shells of isotropic or orthotropic materials.

The nonlinearities in a system are represented by the products of dependent variables or their derivatives in the governing equations. One or more such terms in a product, depending upon the degree of nonlinearity, can be expressed in terms of the previously known values with respect to the marching variable.

Let $X(\xi, \tilde{q})Y(\xi, \tilde{q})$ be any typical product term. Then at any step J of load variable $\tilde{q}$, the value of any one of the terms, say Y, may be expressed in a Taylor series as:

$$(Y)^{(J)}=(Y)^{(J-1)}+\left(\frac{\partial Y}{\partial\tilde{q}}\right)^{(J-1)}\Delta\tilde{q}+\left(\frac{\partial^2 Y}{\partial\tilde{q}^2}\right)^{(J-1)}\frac{\Delta\tilde{q}^2}{2}+\cdots. \quad (20)$$

Retaining only the first three terms in the above series and using a finite difference scheme for expressing the derivatives therein, the product term XY at step J is written as

$$XY = X^J Y^J = X^J Y^* = (AY^{J-1} + BY^{J-2} + CY^{J-3})X^J, \tag{21}$$

where

$$\begin{aligned} &\text{for } J = 1, \quad A = B = C = 0, \\ &\text{for } J = 2, \quad A = 2, \quad B = C = 0, \\ &\text{for } J = 3, \quad A = -2, \quad B = 2.5, \quad C = 0, \\ &\text{for } J > 3, \quad A = 2.5, \quad B = -2, \quad C = 0.5. \end{aligned} \tag{22}$$

The superscript * of Y_J in eqn (21) designates that, during linearization, this term is computed as indicated.

The governing differential equations of equilibrium (18) can now be written in linearized form. In this investigation only spherical shells subjected to uniform external pressure q are considered. Introducing the following nondimensional quantities:

$$\begin{aligned} &\bar{\phi} = \phi_1 - \phi_0, \quad l = R\bar{\phi}, \quad \xi = (s - R\phi_0)/l, \quad t = h/l, \\ &(\bar{u}\,\bar{w}\,\bar{\alpha}) = \frac{1}{h^2}(\hat{u}l\,\hat{w}h\,\hat{\alpha}lh), \quad a_{44} = K^2 A_{44}/E_L h, \\ &(a_{ij}\,b_{ij}\,d_{ij}) = \frac{1}{E_L h^3}(A_{ij}h^2\,B_{ij}h\,D_{ij}) \quad (i,j = 1,2), \\ &(\bar{N}_s \bar{N}_\theta \bar{Q}_s) = \frac{h^3}{\Delta q_0 l^4}(N_s N_\theta Q_s), \\ &(\bar{M}_s \bar{M}_\theta) = \frac{h^2}{\Delta q_0 l^4}(M_s M_\theta), \end{aligned} \tag{23}$$

where Δq_0 = incremental pressure applied during the load step. Defining

$$y = \bar{\phi} t \quad \text{and} \quad x = \bar{\phi}\cot\phi,$$

the equilibrium equations (18) at the Jth load step are written in nondimensional form as:

$$\begin{aligned} &\bar{N}^J_{s,\xi} + x(\bar{N}^J_s - \bar{N}^J_\theta) + \bar{\phi}\bar{Q}^J_s = I_1[y^2\bar{N}^*_s\bar{u}^J - y\bar{N}^*_s\bar{w}^J_{,\xi} + y^3\bar{M}^*_s\bar{u}^J - y^2\bar{M}^*_s\bar{w}^J_{,\xi} - y^2\bar{M}^*_s\bar{u}^J] \\ &\qquad\qquad - I_2(tx\bar{Q}^*_s\bar{\alpha}^J + t\bar{Q}^*_{s,\xi}\bar{\alpha}^J + t\bar{Q}^*_s\bar{\alpha}^J_{,\xi}), \\ &\bar{M}^J_{s,\xi} + x(\bar{M}^J_s - \bar{M}^J_\theta) - \bar{Q}^J_s/t = I_1[y^2\bar{M}^*_s\bar{u}^J - y\bar{M}^*_s\bar{w}^J_{,\xi}] + I_2\bar{Q}^*_s(t\bar{u}^J_{,\xi} + \bar{\phi}\bar{w}^J), \\ &\bar{\phi}(\bar{N}^J_s + \bar{N}^J_\theta) - x\bar{Q}^J_s - \bar{Q}^J_{s,\xi} = Jt^3 + I_1[t\bar{N}^*_s(\bar{w}^J_{,\xi\xi} - y\bar{u}^J_{,\xi}) \\ &\qquad + tx\bar{N}^*_s(\bar{w}^J_{,\xi} - y\bar{u}^J) + t\bar{N}^*_{s,\xi}(\bar{w}^J_{,\xi} - y\bar{u}^J) - ty\bar{M}^*_s(\bar{w}^J_{,\xi\xi} - y\bar{u}^J_{,\xi} + \bar{\alpha}^J_{,\xi}) \\ &\qquad - txy\bar{M}^*_s(\bar{w}^J_{,\xi} - y\bar{u}^J + \bar{\alpha}^J) - ty\bar{M}^*_{s,\xi}(\bar{w}^J_{,\xi} - y\bar{u}^J + \bar{\alpha}^J)] - I_2\bar{Q}^*_s y\bar{\alpha}^J. \end{aligned} \tag{24}$$

Substituting for strains in eqns (13) and (14) from eqn (8), and nondimensionalizing using eqns (23), the nonlinear equations expressing relations of the stress resultants and stress couples with reference surface displacements can be written as:

$$\begin{aligned} &a_{12}t^2x\bar{u}^J + I_1(\tfrac{1}{2}a_{11}t^2y^2\bar{u}^* - \tfrac{1}{2}b_{11}t^2y^3\bar{u}^* + b_{11}t^2y^2\bar{w}^*_{,\xi})\bar{u}^J \\ &\quad + a_{11}t^2\bar{u}^J_{,\xi} + y(a_{11} + a_{12})\bar{w}^J + I_1(\tfrac{1}{2}a_{11}t^2\bar{w}^*_{,\xi} - a_{11}t^2y\bar{u}^* - \tfrac{1}{2}b_{11}t^2y\bar{w}^*_{,\xi})\bar{w}^J_{,\xi} \\ &\quad + b_{12}t^2x\bar{\alpha}^J - I_1(b_{11}t^2y\bar{w}^*_{,\xi} - b_{11}t^2y^2\bar{u}^*)\bar{\alpha}^J + b_{11}t^2\bar{\alpha}^J_{,\xi} - \frac{\Delta q_0}{E_L t^4}\bar{N}^J_s = 0, \end{aligned}$$

$$a_{22}t^2x\bar{u}^J+I_1(\tfrac{1}{2}a_{12}t^2y^2\bar{u}^*-\tfrac{1}{2}b_{12}t^2y^3\bar{u}^*+b_{12}t^2y^2\bar{w}^*_{,\xi})\bar{u}^J$$
$$+a_{12}t^2\bar{u}^J_{,\xi}+y(a_{12}+a_{22})\bar{w}^J+I_1(\tfrac{1}{2}a_{12}t^2\bar{w}^*_{,\xi}-a_{12}t^2y\bar{u}^*-\tfrac{1}{2}b_{12}t^2y\bar{w}^*_{,\xi})\bar{w}^J_{,\xi}$$
$$+b_{22}t^2x\bar{\alpha}^J-I_1(b_{12}t^2y\bar{w}^*_{,\xi}-b_{12}t^2y^2\bar{u}^*)\bar{\alpha}^J+b_{12}t^2\bar{\alpha}^J_{,\xi}-\frac{\Delta q_0}{E_Lt^4}\bar{N}^J_\theta=0,$$

$$b_{12}t^2x\bar{u}^J+I_1(\tfrac{1}{2}b_{11}t^2y^2\bar{u}^*-\tfrac{1}{2}d_{11}t^2y^3\bar{u}^*+d_{11}t^2y^2\bar{w}^*_{,\xi})\bar{u}^J$$
$$+b_{11}t^2\bar{u}^J_{,\xi}+y(b_{11}+b_{12})\bar{w}^J+I_1(\tfrac{1}{2}b_{11}t^2\bar{w}^*_{,\xi}-b_{11}t^2y\bar{u}^*-\tfrac{1}{2}d_{11}t^2y\bar{w}^*_{,\xi})\bar{w}^J_{,\xi}$$
$$+d_{12}t^2x\bar{\alpha}^J-I_1(d_{11}t^2y\bar{w}^*_{,\xi}-d_{11}t^2y^2\bar{u}^*)\bar{\alpha}^J+d_{11}t^2\bar{\alpha}^J_{,\xi}-\frac{\Delta q_0}{E_Lt^4}\bar{M}^J_s=0,$$

$$b_{22}t^2x\bar{u}^J+I_1(\tfrac{1}{2}b_{12}t^2y^2\bar{u}^*-\tfrac{1}{2}d_{12}t^2y^3\bar{u}^*+d_{12}t^2y^2\bar{w}^*_{,\xi})\bar{u}^J$$
$$+b_{12}t^2\bar{u}^J_{,\xi}+y(b_{12}+b_{22})\bar{w}^J+I_1(\tfrac{1}{2}b_{12}t^2\bar{w}^*_{,\xi}-b_{12}t^2y\bar{u}^*-\tfrac{1}{2}d_{12}t^2y\bar{w}^*_{,\xi})\bar{w}^J_{,\xi}$$
$$+d_{22}t^2x\bar{\alpha}^J-I_1(d_{12}t^2y\bar{w}^*_{,\xi}-d_{12}t^2y^2\bar{u}^*)\bar{\alpha}^J+d_{12}t^2\bar{\alpha}^J_{,\xi}-\frac{\Delta q_0}{E_Lt^4}\bar{M}^J_\theta=0,$$

$$a_{44}t\bar{w}^J_{,\xi}-a_{44}t^2\bar{\phi}\bar{u}^J+a_{44}\bar{\alpha}^Jt+I_2(a_{44}t^3\bar{\alpha}^*\bar{u}^J_{,\xi}+a_{44}t^2\bar{\phi}\bar{\alpha}^*\bar{w}^J)-\frac{\Delta q_0}{E_Lt^4}\bar{Q}^J_s=0. \tag{25}$$

BOUNDARY CONDITIONS

In the present analysis, the following boundary conditions are considered:

(a) Immovable clamped edges at $\xi = 0$ or $\xi = 1$,

$$\bar{w}=\bar{u}=\bar{\alpha}=0; \tag{26a}$$

(b) Regularity conditions at the apex of a shell closed at a pole

$$\bar{w}_{,\xi}=\bar{u}=\bar{\alpha}=0. \tag{26b}$$

CHEBYSHEV SERIES

In the present analysis, the analytical solution of the system of differential equations in (24), (25) with the associated boundary conditions in (26) is sought using the Chebyshev–Galerkin spectral method. The various properties of Chebyshev polynomials which are important and are employed here are discussed below.

Any continuous function $f(\xi)$ in the interval $0 \leqslant \xi \leqslant 1$ can be represented by a series of the form,

$$f(\xi)=\frac{a_0}{2}T_0^*(\xi)+\sum_{r=1}^{\infty}a_rT_r^*(\xi), \tag{27}$$

where a_r ($r = 0, 1, 2, 3, \ldots$) are the constants to be determined so as to obtain the best possible fit.

Here,

$$T_r^*(\xi) = r\text{th polynomial in the shifted Chebyshev polynomial series,}$$
$$= \cos rt \quad \text{where} \quad \cos t = (2\xi-1). \tag{28}$$

These shifted Chebyshev polynomials satisfy the recurrence relations

$$T^*_{r+1}(\xi)+T^*_{r-1}(\xi) = 2(2\xi-1)T^*_r(\xi) \tag{29}$$

and the orthogonality conditions

$$\int_0^1 \frac{T^*_m(\xi)T^*_n(\xi)}{\sqrt{\xi}\sqrt{(1-\xi)}}\mathrm{d}\xi = \begin{cases} 0 & \text{for } m \neq n, \\ \pi/2 & \text{for } m = n \neq 0, \\ \pi & \text{for } m = n = 0. \end{cases} \tag{30}$$

For any continuous function $f(\xi)$, the series (27) is fast converging and good approximation is obtained by taking a finite number of terms in the above series. $f(\xi)$ can be expressed as

$$f(\xi) = \frac{a_0}{2}T^*_0(\xi)+\sum_{r=1}^{N} a_r T^*_r(\xi), \tag{31}$$

where, for a known function $f(\xi)$, the coefficients a_r are given by

$$a_r = \frac{2}{\pi}\int_0^1 \frac{f(\xi)T^*_r(r)}{\sqrt{\xi}\sqrt{(1-\xi)}}\mathrm{d}\xi, \quad (0 \leqslant r \leqslant N). \tag{32}$$

If $f(\xi)$ and $g(\xi)$ are two continuous functions represented by truncated Chebyshev series as

$$f(\xi) = \sum_{r=0}^{M}{}^{+} a_r T^*_r(\xi) \quad \text{and} \quad g(\xi) = \sum_{r=0}^{N}{}^{+} b_r T^*_r(\xi), \tag{33}$$

where $^+$ indicates that the first term of these series must be halved, then the product of these functions can be written in the Chebyshev series form as

$$f(\xi)g(\xi) = \sum_{r=0}^{M+N}{}^{+} c_r T^*_r(\xi), \tag{34}$$

where

$$c_0 = \sum_{i=0}^{M+N}{}^{+} a_i b_i,$$

$$c_r = \tfrac{1}{2}\sum_{i=0}^{M+N} a_i(b_{(i+r)}+b_{|i-r|}) \quad \text{for } 1 \leqslant r \leqslant M+N. \tag{35}$$

If $f(\xi)$ and $f'(\xi)$ are expressed in Chebyshev series as

$$f(\xi) = \sum_{r=0}^{N}{}^{+} a_r T^*_r(\xi) \quad \text{and} \quad f'(\xi) = \sum_{r=0}^{N-1}{}^{+} a_r^{(1)} T^*_r(\xi),$$

the coefficients satisfy the recursive relations

$$a^{(1)}_{r-1} - a^{(1)}_{r+1} = 4ra_r \tag{36}$$

and $a_r^{(1)}$ can be obtained using a_r as (Karageorghis, 1988)

$$c_r a_r^{(1)} = 4\sum_{j=1}^{N}(r+2j-1)a_{(r+2j-1)}, \tag{37}$$

where

$$c_0 = 2, \quad c_r = \begin{cases} 1 & r > 1, \\ 0 & r < 1. \end{cases} \tag{37a}$$

SOLUTION PROCEDURES

The three displacement components and the five stress resultants occurring in the governing equations (24)–(26) at load step J are expressed in Chebyshev series as

$$(\bar{\alpha}\bar{u}\bar{w})^J = \sum_{r=0}^{N}{}^{+} (\alpha_r u_r w_r)^J T_r^*(\xi),$$
$$(\bar{N}_s \bar{N}_\theta \bar{Q}_s \bar{M}_s \bar{M}_\theta)^J = \sum_{r=0}^{N-1}{}^{+} (A_r B_r C_r D_r E_r)^J T_r^*(\xi). \tag{38}$$

There are $(8N+3)$ coefficients to be determined in (38) for the complete solution at load step J. Six boundary conditions, three at each edge (one edge and the pole in the case of shells with an apex) in (26) make it necessary to set up $(8N-3)$ algebraic equations from consideration of (24) and (25).

The trigonometric term cot ϕ in the governing equations is expressed in terms of a Chebyshev series in terms of ξ as

$$\cot \phi = \sum_{r=0}^{M1}{}^{+} X_r T_r^*(\xi). \tag{39}$$

The coefficients X_r can be obtained by forcing the series to take on the actual values at a number of chosen points in $0 < \xi < 1$.

Using the solution methodology presented by Alwar and Narasimhan (1990), the eight equations in (24) and (25) are written in terms of Chebyshev polynomials. Equating the coefficients of Chebyshev polynomials of the same degree r on either side of equations for $0 \leqslant r \leqslant N-2$ in respect of the equilibrium equations (24) and for $0 \leqslant r \leqslant N-1$ for stress resultant–displacement equations (25), $8n-3$ algebraic equations can be written in terms of coefficients in (38). With the six boundary conditions, there are $8n+3$ equations for as many unknowns. These $8n+3$ equations can be written in matrix form as:

$$\begin{bmatrix} [Z1] & [Z2] \\ [Z3] & [Z4] \end{bmatrix} \begin{Bmatrix} \{R_1\} \\ \{R_2\} \end{Bmatrix} = \begin{Bmatrix} \{p\} \\ \{0\} \end{Bmatrix}, \tag{40}$$

where

$$\{R_1\}^T = \{\{\alpha\}^T\{u\}^T\{w\}^T\},$$
$$\{R_2\}^T = \{\{A\}^T\{B\}^T\{C\}^T\{D\}^T\{E\}^T\},$$
$$\{p\}^T = \{\{0\}\{0\}\{q\}^T\}. \tag{41}$$

The vector $\{\alpha\}$ is defined by the coefficients in the series of $\bar{\alpha}$ as

$$\{\alpha\}^T = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_N\}. \tag{42}$$

The vectors $\{u\}$, $\{w\}$, $\{A\}$, $\{B\}$, $\{C\}$, $\{D\}$ and $\{E\}$ are also similarly defined.

The first set of matrix equations in (40) are the algebraic equations due to the boundary conditions and equilibrium equations, while the second set are due to the stress resultant–displacement relations. The authors had observed that about 12–14 terms of a polynomial series are required for getting accurate results in the case of linear analysis of a laminated spherical shell. This leads to a large system of algebraic equations to be solved for each iteration of a load step, which makes the computations costlier in the case of a nonlinear analysis. However, the computational effort can be reduced by resorting to a solution with partitioning. In the system of matrix equations (40), the matrices $[Z2]$ and $[Z4]$ are the same for any load step. Equations (40) can be written as:

$$[[Z1]-[Z2][Z4]^{-1}[Z3]]\{R_1\} = \{p\},$$
$$\{R_2\} = -[Z4]^{-1}[Z3]\{R_1\}. \tag{43}$$

At the first iteration of the first load step, all four matrices $[Z1]$, $[Z2]$, $[Z3]$ and $[Z4]$ and the vector $\{p\}$ are generated. The matrix $[Z4]$ is inverted and is stored along with $[Z2]$ for computations in all further load steps too. Algebraic equations (43) are set up and are solved by Gaussian elimination with pivoting. For all further load steps, only the matrices $[Z1]$ and $[Z3]$ need be generated along with the load vector. Within a load step, equilibrium iterations are performed by taking the current values as those obtained in the preceding iteration. Equilibrium iteration is continued until an average transverse deflection criterion is satisfied wherein the values of average transverse deflection W_{ave} in two successive iterations should not differ by more than a pre-fixed percentage. For all the computations reported herein, a convergence tolerance of 0.05% has been adopted. W_{ave} is defined as

$$W_{\text{ave}} = \frac{2R}{l}\int_0^1 \bar{w}\xi \, d\xi. \tag{44}$$

RESULTS AND DISCUSSIONS

In all the results presented herein, loading due to uniform external pressure has been assumed. The fibre orientations of the layers are specified either as 90° or 0°, depending upon whether the layer is circumferentially reinforced or meridionally reinforced. The material properties along the principal material directions in different layers are assumed to be the same. The thicknesses of all the layers are also assumed to be the same. Shear correction factors calculated based upon Whitney's method (1972) are used throughout the present work.

Convergence studies are conducted to ascertain the number of terms in the Chebyshev series to be used in the analysis. Table 1 presents the results of such an analysis in respect of a two layered annular spherical shell. It can be observed that about 12 terms in the Chebyshev series are sufficient for converged results. However, a 14-term approximation has been used in the parametric studies.

Table 2 presents a comparison between the linear solution for a laminated shell as obtained in the present solution and those presented by Alwar (1990). It can be seen that the results agree well.

Figure 2 presents the results obtained by the present method of solution in respect of isotropic spherical shells for three different shell parameters. To circumvent the problems of convergence, a small hole was assumed at the apex of the shell and the regularity boundary conditions were applied at the inner edge of this small hole. It can be observed that the present solution yields results which are agreeing well with results presented by Nath and Alwar (1978). The small discrepancies between the results are attributed to the fact that Alwar and Nath have used the shallow shell theory for the solution.

The nonlinear responses of clamped two layer cross-ply shell are shown in Fig. 3 for different orthotropy ratios. It is found that for a given shell configuration and lamination sequence, shells with higher orthotropy ratios show lower nonlinearity and have higher critical pressures. The load–deflection curves shown in Fig. 4 depict the effect of the R/h ratio on the large deflection behaviour of two layer cross-ply shells. It is observed that the lower the R/h ratio, because of an increased membrane action, the greater the reduction in the softening nonlinearity.

Typical variations of the meridional stress resultant N_s with increase in applied pressure are shown in Fig. 5. It is seen in all the cases presented that the stress resultant initially

Table 1. Convergence study, $R/h = 20$, $\phi_1 = 60°$, $\phi_0 = 20°$, $E_L/E_T = 20$, $\nu_{LT} = 0.28$, $G_{LT} = 0.5\,E_T$, $G_{TT} = 0.2\,E_T$. Lamination 0°/90°, Boundary conditions: Clamped

	W_{ave}		
ql^4/E_Lh^4	10 terms	12 terms	14 terms
2.0	0.02334	0.02334	0.02334
4.0	0.04730	0.04730	0.04730
6.0	0.07193	0.07193	0.07193
8.0	0.09728	0.09727	0.09728
10.0	0.12340	0.12339	0.12340
12.0	0.15036	0.15035	0.15035
14.0	0.17824	0.17823	0.17824
16.0	0.20712	0.20712	0.20713
18.0	0.23714	0.23713	0.23713
20.0	0.26844	0.26842	0.26843
24.0	0.33575	0.33573	0.33574
28.0	0.41206	0.41204	0.41206
32.0	0.50523	0.50540	0.50536

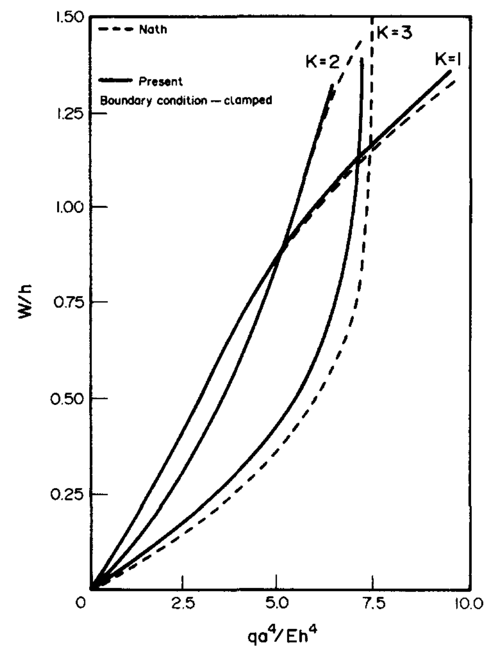

Fig. 2. Load deflection curves for isotropic shells.

Table 2. Linear solution for laminated shell, $R/h = 30$, $\phi_1 = 90°$, $\phi_0 = 10°$, $E_L/E_T = 20$, $\nu_{LT} = 0.28$, $G_{LT} = 0.5\,E_T$, $G_{TT} = 0.2\,E_T$. Lamination 0°/90°. Boundary conditions: Clamped

	Present solution	Alwar and Narasimhan (1990)
1. Maximum deflection (W)	0.14796	0.14782
2. Maximum meridional moment resultant ($\tilde{M}_s$)	1.171	1.175
3. Maximum hoop stress resultant ($\tilde{N}_\theta$)	5.39	5.410

$W = 10^4\,(E_Th^3/ql^4)\hat{w}$ $\quad \tilde{M}_s = 10^6\,(h^2M_s/ql^4)$ $\quad \tilde{N}_\theta = 10^6\,(h^3N_\theta/ql^4)$

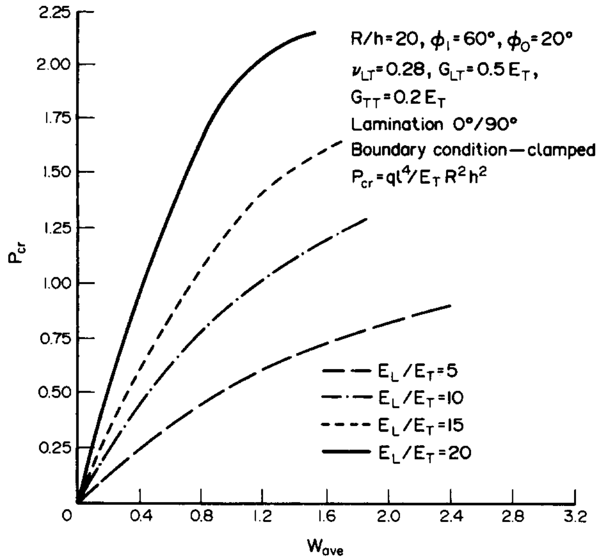

Fig. 3. Effect of orthotropy ratio on nonlinear static response.

increases with pressure, reaches a maximum and then starts decreasing, the peak values being higher for laminates with more layers.

The effect of shear deformation on the response of a two layered laminated spherical shells has been studied and the results have been presented in Figs 6 and 7. The results corresponding to the case with no shear deformations (Classical Theory) were obtained by setting very high values for transverse shear moduli ($G_{TT} = 10000\, E_T$). While the maximum deflection is overestimated by the Classical Theory (Fig. 6), the average deflection, which is a more general index of the total stiffness of the structure, is underestimated by the Classical Theory (Fig. 7). These observations are again in conformity with the authors' observations in earlier investigations (Alwar and Narasimhan, 1990; Narasimhan and Alwar, 1992) that the Classical Theory overestimates both the maximum deflection and natural frequencies of laminated orthotropic cross-ply spherical shells.

Figures 8 and 9 present the effect of different nonlinear terms considered in the analysis on the predicted nonlinear response. The three cases investigated are:

(a) Von Karman rotation term only is considered ($I_1 = 1$, $I_2 = I_3 = 0$).

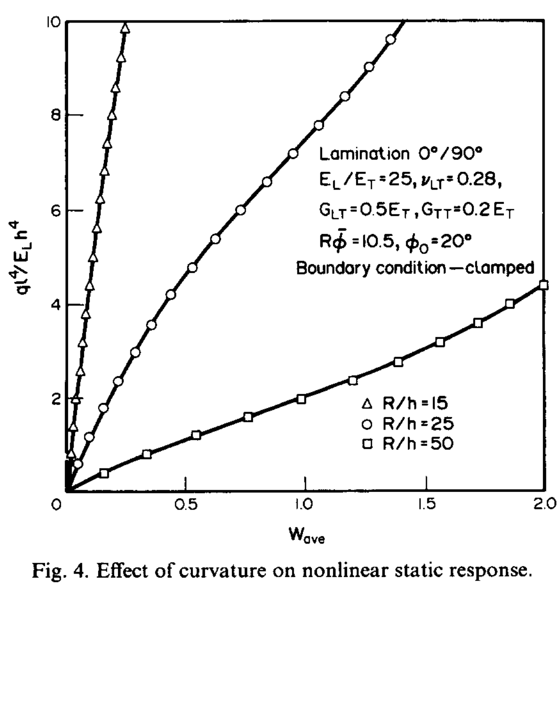

Fig. 4. Effect of curvature on nonlinear static response.

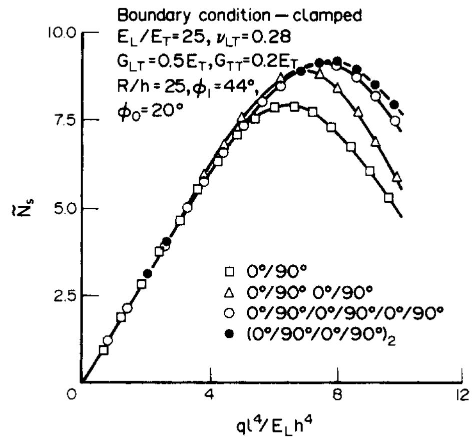

Fig. 5. Variation of meridional stress resultant N_s with applied pressure ($\tilde{N}_s = 10^6h^3N_s/E_Ll^4$).

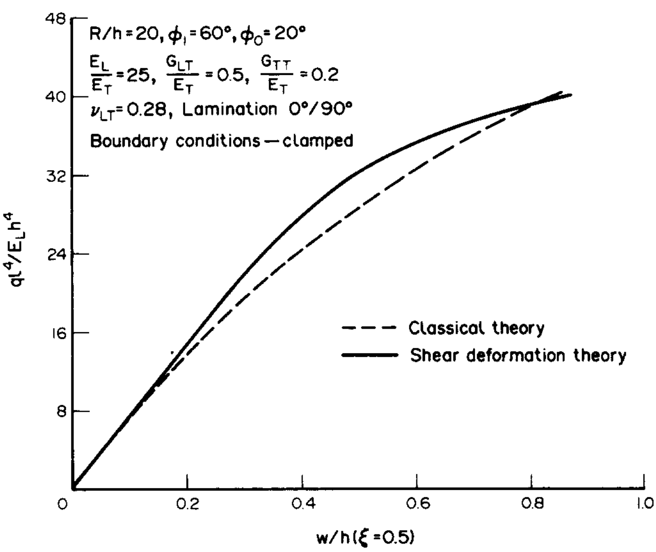

Fig. 6. Effect of shear deformation on nonlinear static response.

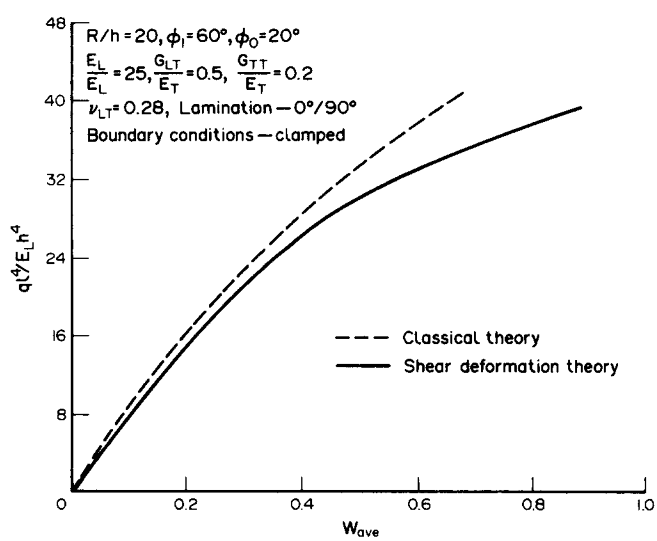

Fig. 7. Effect of shear deformation on nonlinear static response.

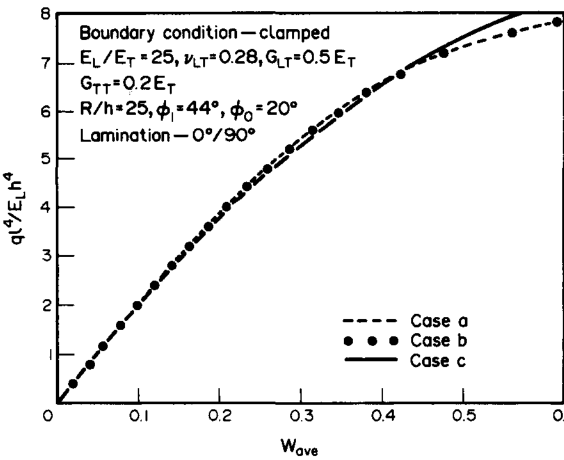

Fig. 8. Effect of different ranges of nonlinearities on the nonlinear static response of laminated shell.

(b) Only the simplified Von Karman rotation term is considered. The Von Karman rotation is given by $(\mathrm{d}\hat{w}/\mathrm{d}s-\hat{u}/R)$. When the shell is thin, $\hat{u}/R \ll \mathrm{d}\hat{w}/\mathrm{d}s$, and hence can be neglected in comparison to $\mathrm{d}\hat{w}/\mathrm{d}s$ in the rotation term $(I_2 = I_3 = 0)$.

(c) The product of strains with rotations are also considered $(I_1 = I_2 = 1, I_3 = 0)$.

It can be seen from Fig. 8 that, when only the Von Karman rotations are considered—as applicable either to a deep shell (case a) or to a shallow shell (case b), the nonlinearity is overestimated when compared to case c. Also, cases a and b give almost identical results for the shells considered herein. In Fig. 9 it can be observed that the Von Karman type theories underestimate the peak values of the meridional stress resultant.

CONCLUSIONS

The problem of an orthotropic laminated spherical shell undergoing large axisymmetric deformations is formulated using the principle of virtual work. An analytical solution based on the Chebyshev–Galerkin spectral method is proposed. Numerical results presented indicate that the method of solution gives sufficiently accurate results.

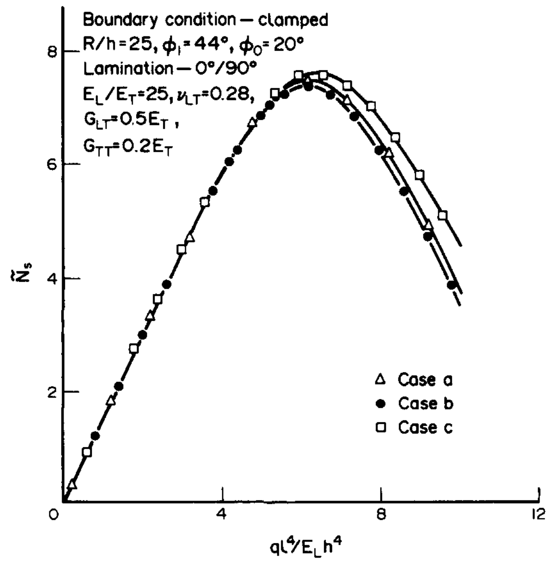

Fig. 9. Effect of different ranges of nonlinearities on the large deflection behaviour of laminated shells.

The effects of various geometric and material parameters on the nonlinear axisymmetric response of laminated annular shells subjected to uniform external pressure have been studied. It is observed that softening nonlinearity of these shells increases with an increase in the R/h ratio but decreases with an increase in orthotropy ratio and number of layers. The transverse shear deformation has an appreciable effect on the response of laminated spherical shells and usually increases the maximum average deflection W_{ave} which is not so in the case of maximum deflections. It is also observed that Von Karman type theories overestimate the softening nonlinearity of static response of laminated spherical shells.

REFERENCES

Alwar, R. S. and Narasimhan, M. C. (1990). Application of Chebyshev polynomials to the analysis of laminated axisymmetric spherical shells. *Compos. Struct.* **15,** 215–237.

Alwar, R. S. and Nath, Y. (1977). Nonlinear dynamic response of circular plates subjected to transient loads. *Franklin Inst. Jl* **303,** 521–542.

Booton, M. and Tennyson, R. C. (1979). Buckling of imperfect anisotropic circular cylindrical shells under combined loadings *AIAA Jl* **17,** 276–287.

Chang, T. Y. and Sawamiphakdi, K. (1981). Large deformation analysis of laminated shells by finite element method. *Comput. Struct.* **13,** 331–340.

Chao, W. C. and Reddy, J. N. (1984). Analysis of laminated composite shells using a degenerated 3-D element. *Int. J. Num. Meth. Engng* **20,** 1991–2007.

Dennis, S. T. and Palazotto, A. N. (1990). Large displacement and rotation formulation for laminated shells including parabolic transverse shear. *Int. J. Nonlin. Mech.* **25,** 67–85.

Karageorghis, A. (1988). A note on the Chebyshev coefficients of the general order derivative of an infinitely differentiable function. *J. Comput. Appl. Mech.* **21,** 129–132.

Librescu, L. (1987). Refined geometrically nonlinear theories of anisotropic laminated shells. *Q. Appl. Math.* **45,** 1–27.

Narasimhan, M. C. and Alwar, R. S. (1992). Free vibration analysis of laminated orthotropic spherical shells. *J. Sound. Vibr.* **154,** 519–529.

Nath, Y. and Alwar, R. S. (1978). Nonlinear static and dynamic response of spherical shells. *Int. J. Nonlin. Mech.* **13,** 157–170.

Noor, A. K. and Anderson, C. M. (1982). Mixed models and reduced/selective integration displacement models for nonlinear shell analysis. *Int. J. Num. Meth. Engng* **18,** 1429–1454.

Noor, A. K. and Heartly, S. J. (1977). Nonlinear shell analysis via the mixed isoparametric elements. *Comput. Struct.* **7,** 615–626.

Noor, A. K. and Mathers, M. D. (1974). Nonlinear finite element analysis of laminated composite shells. *Proc. Int. Conf. on Computational Methods in Nonlinear Mechanics*. Austin, Texas.

Noor, A. K. and Peters, J. M. (1986). Nonlinear analysis of anisotropic panels. *AIAA Jl* **24,** 1543–1553.

Novozhilov, V. V. (1961). *Theory of Elasticity*. (Translated by J. K. Lusker) Pergamon Press, Oxford.

Reddy, J. N. and Chandrasekhara, K. (1985). Nonlinear analysis of laminated shells including transverse shear strains. *AIAA Jl* **23,** 440–441.

Saigal, S., Kapania, R. K. and Yang, T. Y. (1986). Geometrically nonlinear finite element analysis of imperfect laminated shells. *J. Comp. Mater.* **20,** 197–213.

Sheinman, I. and Simitses, G. J. (1983). Buckling and post-buckling of imperfect cylindrical shells under axial compression. *Comput. Struct.* **17,** 277–285.

Stein, M. (1986). Nonlinear theory for plates and shells including effects of transverse shearing. *AIAA Jl* **24,** 1537–1544.

Whitney, J. M. (1972). Stress analysis of thick laminated composite and sandwich plates. *J. Comp. Mater.* **6,** 426–440.